Adding and multiplying random matrices: A generalization of Voiculescu's formulas

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In this paper, we give an elementary proof of the additivity of the functional inverses of the resolvents of large N random matrices, using recently developed matrix model techniques. This proof also gives a very natural generalization of these formulas to the case of measures with an external field. A similar approach yields a relation of the same type for multiplication of random matrices. [S1063-651X(99)02305-3]

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I. INTRODUCTION

In the theory of free random variables [1], a remarkable additivity property of the functional inverses of the spectral resolvents is found, allowing the addition of random variables. There is also a similar formula for multiplication of random variables. From now on, we shall call them Voiculescu's formulas. These mathematical results have some interesting applications: indeed, it turns out that large size (independent) random matrices with certain measures are free variables. Therefore it becomes possible to compute the resolvent of the sum of two random matrices from the knowledge of the resolvent of the separate matrices, i.e., to add (and multiply) large random matrices. This, in turn, applies to various physical situations: ''deterministic +random'' problem [2,3] (noting that for Gaussian randomness, the addition formula essentially reduces to Pastur's equation [4]), random matrix methods applied to QCD [5], non-Hermitean random matrices [6], and the Anderson model [7]. Since in the ''planar'' large N limit (N size of the matrices) that we consider here one cannot compute n-point connected correlations of the eigenvalues or other $N\to\infty$ subdominant corrections, alternative methods (such as the supersymmetric method, see review [8] and references therein) may be required for a more detailed analysis; but for many problems, it is still very important to be able to compute the density of eigenvalues (one-point function), which Voiculescu's formulas provide.

It is therefore of great interest to find an elementary proof of these formulas. We shall mention one such proof by Zee [3] of the addition formula, which is based on a perturbative approach: the measures of two random Hermitean matrices M_1 and M_2 are assumed to be derived from an action of the form $\operatorname{tr} V(M)$, where V is a polynomial, and the perturbative expansion is represented diagrammatically, leading to a diagrammatic proof of Voiculescu's formula.

However, this proof has limitations. First it assumes *U(N)* invariance of the actions. Of course one might object that if we assume both measures to be non-*U(N)*-invariant, then Voiculescu's formula is not true any more (there is an obvious counterexample, which is the case of two fixed matrices). And if one measure is *U(N)*-invariant and the other is not, one can freely replace the noninvariant measure $d\mu(M)$ with an invariant one by averaging on the unitary group:

$$d\mu_{\text{eff}}(M)=\int_{\Omega\in U(N)}d\Omega\ d\mu(\Omega M\Omega^{\dagger}).$$

This replacement will not affect the resolvent of the sum of M and of another random matrix with *U(N)*-invariant measure; however, it is not completely innocent since even if the original measure $d\mu(M)$ was derived from a simple polynomial action, there is no reason for $d\mu_{\text{eff}}(M)$ to possess the same property.

We see that the problem is that this proof does not allow for general enough measures; in particular, a very interesting physical application is the case of a fixed matrix (for the deterministic+random problem), where the corresponding measure is highly singular (δ function) and does not fit in this perturbative framework.

We propose in this paper a proof of both addition and multiplication formulas, which makes very few assumptions on the measures; it is based on recently developed matrix model techniques [9] which have been successfully applied to physical models [10]. In Sec. II, we shall show how to add matrices by introducing an external field in the measure (as in Ref. [9]), and in Sec. III, we shall multiply matrices by adding this time a character in the measure (as in Ref. [10]). The proof has the obvious advantage that it generalizes the usual addition formula to the case of a measure with an external field (and similarly, the multiplication formula to the case of a measure with a character insertion). Section IV is devoted to a summary of the results and conclusions.

II. ADDING RANDOM MATRICES

Before addressing the problem of the addition of several matrices, we shall explain our approach by considering a single $N\times N$ Hermitean matrix M with a *U(N)*-invariant measure $d\mu(M)$. The only assumption we make about this measure is that the diagonalization of M leads to a saddle point for the eigenvalues of M; that is, after integrating out the angular degrees of freedom, the dominant large N contribution is obtained by simply considering the eigenvalues to be fixed (up to a permutation of the eigenvalues). This is a reasonable assumption, since as $N\to\infty$, there are only N eigenvalues, as opposed to the N^2 degrees of freedom of the full matrix. For example, a typical measure that is encountered in physical problems is

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$$d\mu(M)=\prod_i dM_{ii}\prod_{i<j} d\ \mathrm{Re}\,M_{ij}d\ \mathrm{Im}\,M_{ij}\exp(-S(M)), \tag{2.1}$$

where *S(M)* is the action, which is invariant— $S(M)=S(\Omega M\Omega^\dagger)$ for all $\Omega\in U(N)$—and scales likes N^2 as $N\to\infty$, which ensures a saddle point for the eigenvalues [for example, *S(M)* can be of the form $S(M)=N\ \mathrm{tr}\,V(M)$, where *V* is a given polynomial, but more general actions with products of traces are possible]. However, the action *dμ(M)* does not have to be of the form (2.1), and in particular can be more singular [for example, for a fixed matrix, after averaging over the unitary group *U(N)*, the measure is a δ function for the eigenvalues].

We now introduce the partition function with an additional external field *A* (see Refs. [11], [12], and [9] for the appearance of such an external field in physical models):

$$Z(A)=\int d\mu(M)\exp(N\ \mathrm{tr}\,MA), \tag{2.2}$$

where *A* is a fixed Hermitean matrix. When $N\to\infty$, one must consider a sequence of $N\times N$ matrices *A* such that their spectral density tends to a continuous density $\rho_A(a)$ on the real axis. Since the measure is *U(N)*-invariant, *Z(A)* depends only on the eigenvalues of *A*, and for definiteness we shall choose *A* to be diagonal, with eigenvalues a_j, $j=1,\ldots,N$.

We can go over to the eigenvalues λ_i of *M* by using the Itzykson-Zuber-Harish Chandra formula [13]:

$$Z[a_j]=\int d\mu[\lambda_i]\frac{\det[\exp(N\lambda_i a_j)]}{\Delta[\lambda_i]\Delta[a_j]}, \tag{2.3}$$

where $\Delta[\ ]$ is the Van der Monde determinant, and $d\mu[\lambda_i]$ is the resulting measure on the eigenvalues; for example, with a measure of the type (2.1), we have

$$Z[a_j]=\int\prod_i d\lambda_i\exp(-S[\lambda_i])\Delta[\lambda_i]\frac{\det[\exp(N\lambda_i a_j)]}{\Delta[a_j]}, \tag{2.4}$$

where we have used the fact that the action *S* only depends on the eigenvalues λ_i of *M*.

Finally we introduce the logarithmic derivative of *Z* with respect to the eigenvalues a_j. According to Eq. (2.2), it is simply expressed as an average,

$$\frac{1}{N}\frac{\partial}{\partial a_j}\ln Z[a_j]=\langle M_{jj}\rangle_A,$$

where the subscript *A* indicates that the average is made in the presence of the external field, i.e., with the measure $d\mu(M)\exp(N\ \mathrm{tr}\,MA)$. A more useful expression for this logarithmic derivative is found by applying Eq. (2.3):

$$\frac{1}{N}\frac{\partial}{\partial a_j}\ln Z[a_j]=\frac{1}{N}\left\langle\frac{\partial}{\partial a_j}\ln\frac{\det[\exp(N\lambda_i a_j)]}{\Delta[a_j]}\right\rangle_A. \tag{2.5}$$

The kind of derivative that appears in Eq. (2.5) has been studied in Ref. [9]; we shall briefly review the results we need, and refer the reader to the appendix 1 of Ref. [9] for the technical details. In the large *N* limit, the spectral density of *A* tends by definition to the continuous density ρ_A, and similarly, since there is a saddle point for the eigenvalues of *M*, we assume that the spectral density ρ_M of *M* becomes also continuous. Then, the derivative with respect to a_j [Eq. (2.5)] becomes an analytic function $f(a_j)$ of its argument a_j, of the form

$$f(a)=\lambda(a)-\omega_A(a). \tag{2.6}$$

Let us define the two functions in Eq. (2.6). $\omega_A(a)$ is the resolvent of *A*:

$$\omega_A(a)=\frac{1}{N}\mathrm{tr}\frac{1}{a-A}=\int\frac{da'\rho_A(a')}{a-a'}.$$

It is an analytic function of *a* except for a cut on the support of *A* (which is contained in the real axis). If we introduce the notation $\bar{\omega}_A(a)=\frac{1}{2}[\omega_A(a+i0)+\omega_A(a-i0)]$ for *a* real, so that $\omega_A(a\pm i0)=\bar{\omega}_A(a)\mp i\pi\rho_A(a)$, then

$$\bar{\omega}_A(a_j)=\frac{1}{N}\frac{\partial}{\partial a_j}\ln\Delta[a_j]. \tag{2.7}$$

Similarly, *λ(a)* is defined by the following requirements: it has the same cut as $\omega_A(a)$ on the support of ρ_A, and it satisfies

$$\bar{\lambda}(a_j)=\frac{1}{N}\left\langle\frac{\partial}{\partial a_j}\ln\ \det[\exp(\lambda_i a_j)]\right\rangle_A. \tag{2.8}$$

Of course, *λ(a)* may have more cuts then $\omega_A(a)$, whose positions are left undefined; so one should really think of *λ(a)* as a multivalued function, living on a branched covering of the complex plane.

Note that combining Eqs. (2.7) and (2.8) and using the fact that $\omega_A(a)$ and *λ(a)* have the same cut, one finds the expression (2.6) for the logarithmic derivative (2.5).

It is now possible to connect the function *λ(a)* with the resolvent $\omega_M(\lambda)$ of *M*:

$$\omega_M(\lambda)=\left\langle\frac{1}{N}\mathrm{tr}\frac{1}{\lambda-M}\right\rangle_A=\int\frac{d\lambda'\rho_M(\lambda')}{\lambda-\lambda'}.$$

Indeed, it was shown in Ref. [9] (see also the earlier work [14]) that if one introduces in a symmetric way the function *a(λ)* with the same cut as $\omega_M(\lambda)$ and such that

$$\bar{a}(\lambda_i)=\frac{1}{N}\left\langle\frac{\partial}{\partial\lambda_i}\ln\ \det[\exp(N\lambda_i a_j)]\right\rangle_A,$$

then *a(λ)* and *λ(a)* are functional inverses of each other as multivalued analytic functions.

Let us now take the limit $A\to 0$ [note that if one directly takes $A=0$, expressions such as Eqs. (2.7) and (2.8) become meaningless; so one must consider a limit where the support of ρ_A has a finite size but becomes smaller and smaller], which is $\rho_A(a)\to\delta(a)$ or still $\omega_A(a)\to 1/a$. In this limit, from the Itzykson–Zuber–Harish Chandra formula, one infers that $\det[\exp(N\lambda_i a_j)]/\Delta[a_j]\sim\Delta[\lambda_i]$, so that $a(\lambda)$ tends to the resolvent $\omega_M(\lambda)$. Therefore, for $A=0$, $\lambda(a)$ is precisely the functional inverse of the resolvent we were looking for.

It is now clear that the obvious factorization property

$$\exp(N\ \mathrm{tr}(M_1+M_2)A)=\exp(N\ \mathrm{tr}\, M_1 A)\exp(N\ \mathrm{tr}\, M_2 A)$$

implies the additivity of the average of its logarithmic derivative:

$$\langle (M_1+M_2)_{jj}\rangle_A=\langle M_{1;jj}\rangle_A+\langle M_{2;jj}\rangle_A .$$

On condition that the two matrices M_1 and M_2 are independent, this can be rewritten as the additivity of the function

$$\lambda(a)-\omega_A(a)$$

or for the particular case $A=0$:

$$\lambda(a)-\frac{1}{a}.$$

This is the essence of Voiculescu's formula for adding random matrices.

Let us see how this works more explicitly, by considering two independent random matrices M_1 and M_2 with measures $d\mu_1(M_1)$ and $d\mu_2(M_2)$. We shall assume both measures to be *U(N)*-invariant, even though, as explained in the Introduction, it is not more difficult to prove the formula with only one *U(N)*-invariant measure and a noninvariant one. Both measures are such that there exists a saddle point for the eigenvalues of M_1 and M_2.

Then one introduces the partition function with an external field:

$$Z(A)=\int\int d\mu_1(M_1)d\mu_2(M_2)\exp(N\ \mathrm{tr}(M_1+M_2)A). \tag{2.9}$$

Again, due to *U(N)* invariance of both measures, *Z(A)* depends only on the eigenvalues a_j of *A*. Therefore we can write that

$$Z[a_j]=\int_{\Omega\in U(N)} d\Omega\ Z(\Omega A\Omega^\dagger), \tag{2.10}$$

where we use the normalized Haar measure on *U(N)*. By performing explicitly the integration over Ω (once more, the Itzykson–Zuber–Harish Chandra integral), we immediately obtain that

$$Z[a_j]=\int\int d\mu_1(M_1)d\mu_2(M_2)\frac{\det[\exp(N\lambda_i a_j)]}{\Delta[\lambda_i]\Delta[a_j]},$$

where the λ_i are the eigenvalues of M_1+M_2. We can now introduce the usual logarithmic derivative with respect to a_j, which is of the form

$$\frac{1}{N}\frac{\partial}{\partial a_j}\ln Z[a_j]=\frac{1}{N}\left\langle \frac{\partial}{\partial a_j}\ln\frac{\det[\exp(N\lambda_i a_j)]}{\Delta[a_j]}\right\rangle_A$$
$$=\lambda(a_j)-\omega_A(a_j), \tag{2.11}$$

where $\lambda(a)$ is connected with the matrix M_1+M_2; in particular, for $A=0$, it is the functional inverse of the resolvent $\omega_{M_1+M_2}(\lambda)$.

On the other hand, one can diagonalize separately M_1 and M_2, since the partition function completely factorizes as $Z(A)=Z_1(A)Z_2(A)$, with obvious notations. One finds

$$Z[a_j]=\int\int d\mu_1(M_1)d\mu_2(M_2)\frac{\det[\exp(N\lambda_{1;i}a_j)]}{\Delta[\lambda_{1;i}]\Delta[a_j]}$$
$$\times\frac{\det[\exp(N\lambda_{2;i}a_j)]}{\Delta[\lambda_{2;i}]\Delta[a_j]}.$$

Therefore,

$$\frac{1}{N}\frac{\partial}{\partial a_j}\ln Z[a_j]=[\lambda_1(a_j)-\omega_A(a_j)]+[\lambda_2(a_j)-\omega_A(a_j)]. \tag{2.12}$$

Combining Eqs. (2.11) and (2.12), we find that the relation

$$\lambda_1(a)+\lambda_2(a)=\lambda(a)+\omega_A(a) \tag{2.13}$$

is valid on the support of the density of $\rho_A(a)$, and by analytic continuation is therefore valid on the whole complex plane.

For $A=0$, the functions $\lambda(a),\lambda_1(a),\lambda_2(a)$ are functional inverses of the corresponding resolvents, and $\omega_A(a)=1/a$, so that Eq. (2.13) reduces to Voiculescu's formula for adding free variables. However, the relation (2.13) still holds for arbitrary *A*, thus generalizing Voiculescu's formula in a highly nontrivial way.

Remarks

(i) If we assume that only one measure [e.g., $d\mu_1(M_1)$] is *U(N)*-invariant, then *Z(A)* no longer depends only on the eigenvalues of *A*; but we can take Eq. (2.10) as a definition of $Z[a_j]$, and then the rest of the proof works identically (except that instead of diagonalizing M_2, one integrates over Ω).

(ii) In the $A=0$ case, the connection to the usual diagrammatic interpretation is the following. One can show that for $A=0$, our definition of $f(a)=\lambda(a)-1/a$ is equivalent to $f(a)\equiv(1/N)(d/da)\ln\langle\exp(NaM_{11})\rangle$, where M_{11} is an arbitrarily chosen diagonal element. $\langle\exp(NaM_{11})\rangle$ being a generating function of the moments of M_{11}, its logarithm generates the connected moments: $\ln\langle\exp(NaM_{11})\rangle=\sum_{n=0}^{\infty}(N^n/n!)a^n\langle M_{11}^n\rangle_c$. Furthermore, using *U(N)* invariance of the measure and both *planarity* and *connectedness* of the diagrams that appear in the perturbative expansion, one has the following large *N* equality:

$$\langle M_{11}^n\rangle_c\overset{N\to\infty}{\sim}\frac{(n-1)!}{N^n}\langle \mathrm{tr}\, M^n\rangle_c .$$

Therefore one finds the usual expansion $f(a)=\sum_{n=0}^{\infty}a^n(1/N)\langle\mathrm{tr}\, M^{n+1}\rangle_c$.

III. MULTIPLYING RANDOM MATRICES

The same type of argument applies to the multiplication of random matrices. Let us start again with a single Hermitean random matrix with a measure $d\mu(M)$ which leads to a saddle point on the eigenvalues. We define the partition function *with a character*:

$$Z[H]=\frac{1}{\dim H}\int d\mu(M)\chi_H(M). \tag{3.1}$$

Here H is a (holomorphic) irreducible representation of $GL(N)$; it can be parametrized in the following way: $H=\{h_j;\ j=1,\dots,N\}$, where the h_j, $j=1,\dots,N$, which form a decreasing sequence of integers, are the *shifted highest weights* of H (the shifted highest weights h_j are connected with the usual highest weights m_j by the formula $h_j=N-j+m_j$). $\chi_H(M)$ is the character of H taken at M. Using Weyl's formula for the character $\chi_H(M)$and the fact that $\dim H=\mathrm{cst}\,\Delta[h_j]$, we can rewrite $Z[H]$ in terms of the h_j:

$$Z[h_j]=\int d\mu(M)\frac{\det[\lambda_i^{h_j}]}{\Delta[\lambda_i]\Delta[h_j]}. \tag{3.2}$$

This expression is very similar to Eq. (2.3) obtained after use of the Itzykson–Zuber–Harish Chandra formula. It is now clear that the same formalism will apply (see appendix 2 of Refs. [9] and [10] for more details).

In the large N limit, we assume that the h_j/N (note the important rescaling of a factor of N) tend to a continuous density $\rho_H(h)$. We can then consider the h_j/N as continuous real variables, and introduce the logarithmic derivatives

$$\frac{\partial}{\partial h_j}\ln Z[h_j]=\left\langle\frac{\partial}{\partial h_j}\ln\frac{\det[\lambda_i^{h_j}]}{\Delta[h_j]}\right\rangle_H \tag{3.3}$$

($\partial/\partial h_j$ stands for $(1/N)[\partial/\partial(h_j/N)]$).

We are now led to the introduction of two functions: the resolvent $\omega_H(h)$,

$$\omega_H(h)=\int\frac{dh'\rho_H(h')}{h-h'},$$

and the function $L(h)$ which has the same cut as $\omega_H(h)$ and whose mean value on it is

$$\mathcal{L}(h_j/N)=\left\langle\frac{\partial}{\partial h_j}\ln\det[\lambda_i^{h_j}]\right\rangle_H.$$

We finally define $\lambda(h)=\exp L(h)$.

The eigenvalues also have a saddle point density $\rho_M(\lambda)$, with its associated resolvent $\omega_M(\lambda)$, and there is a function $h(\lambda)$ which satisfies

$$\not{h}(\lambda_i)=\frac{1}{N}\left\langle\lambda_i\frac{\partial}{\partial\lambda_i}\ln\det[\lambda_i^{h_j}]\right\rangle_H$$

and which has the same cut as $\lambda\omega_M(\lambda)$. $h(\lambda)$ and $\lambda(h)$ are of course functional inverses of each other. Note that we were forced to introduce an extra factor of λ in the definition of $h(\lambda)$, which is the crucial difference from the preceding section. Indeed, let us now choose H to be the trivial representation, so that $h_i=N-i$, that is,

$$\omega_H(h)=\ln\frac{h}{h-1}.$$

Then $\det[\lambda_i^{h_j}]=\Delta[\lambda_i]$ and therefore $h(\lambda)=\lambda\omega_M(\lambda)$: $\lambda(h)$ is now the functional inverse of λ times the resolvent, and not of the resolvent itself, which is something completely different.

Let us now write down a formula for multiplying two matrices M_1 and M_2, with associated measures $d\mu_1(M_1)$ and $d\mu_2(M_2)$. As before, at least one of the two measures must be $U(N)$-invariant. We introduce the partition function with a character

$$Z(H)=\int\int d\mu_1(M_1)d\mu_2(M_2)\frac{\chi_H(M_1M_2)}{\dim H}. \tag{3.4}$$

Direct application of the previous formalism to the product M_1M_2 leads to

$$\frac{\partial}{\partial h_j}\ln Z[h_j]=\ln\lambda(h_j/N)-\omega_H(h_j/N), \tag{3.5}$$

where $\lambda(h)$ is associated to the product M_1M_2.

On the other hand, since one of the two measures is $U(N)$-invariant, we can write that

$$Z[h_j]=\frac{1}{\dim H}\int_{\Omega\in U(N)}\times d\Omega\int\int d\mu_1(M_1)d\mu_2(M_2)\chi_H(\Omega M_1\Omega^\dagger M_2).$$

Using orthogonality relations for matrix elements of irreducible representations, we can integrate over Ω:

$$Z[h_j]=\int\int d\mu_1(M_1)d\mu_2(M_2)\frac{\chi_H(M_1)}{\dim H}\frac{\chi_H(M_2)}{\dim H}.$$

The logarithmic derivative can now be written as

$$\frac{\partial}{\partial h_j}\ln Z[h_j]=[\ln\lambda_1(h_j/N)-\omega_H(h_j/N)]+[\ln\lambda_2(h_j/N)-\omega_H(h_j/N)], \tag{3.6}$$

where $\lambda_1(h)$ and $\lambda_2(h)$ are the functions associated in the usual way to the matrices M_1 and M_2.

Comparing Eqs. (3.5) and (3.6) and exponentiating the resulting formula, as is more appropriate for multiplying matrices, we find

$$\lambda_1(h)\lambda_2(h)=\lambda(h)\exp(\omega_H(h)), \tag{3.7}$$

which is the multiplicativity of the function $\lambda(h)\exp(-\omega_H(h))$.

If we now restrict ourselves to the case of the trivial representation, $\lambda(h),\lambda_1(h),\lambda_2(h)$ are functional inverses of λ times the corresponding resolvents, and $\omega_H(h)=\ln(h/(h-1))$, so that

$$\lambda_1(h)\lambda_2(h)=\lambda(h)\frac{h}{h-1}. \tag{3.8}$$

Note once more that the functions $\lambda(a)$ in Eq. (2.13) and the functions $\lambda(h)$ in Eq. (3.7) are not directly related to each other since they are expressed in terms of different variables.

IV. CONCLUSION

We have proven two main formulas: Eq. (2.13) for the addition of random matrices, and Eq. (3.7) for their multiplication. As far as the author knows, the second formula, even in its usual form [Eq. (3.8)], does not have a simple diagrammatic proof.

The proofs used above have the advantage that they clearly highlight the key hypothesis needed for the results to hold: (i) *U(N)* invariance of (at least one of) the two measures, and (ii) an analyticity property of the resolvents. Let us discuss these two points.

The *U(N)* invariance of the measure is an essential ingredient of the proof: without it one cannot integrate over the unitary group to use the Itzykson–Zuber–Harish Chandra formula for the orthogonality formula for characters. This is completely consistent with the assertion found in the mathematical literature [15,16] that the two matrices should be independently *U(N)*-rotated with respect to each other in order to ensure freeness. As has already been mentioned, this hypothesis is obviously necessary (as the case of two fixed matrices shows); but let us also note that when one keeps the external field *A* nonzero (or the representation *H* nontrivial), then one obtains addition/multiplication formulas which are *different* from Voiculescu's formulas (and, generically, incompatible with them); so that for these measures [which of course also break *U(N)* invariance], the random matrices are no longer free variables, but still satisfy addition/multiplication formulas.

The analyticity property of the resolvents stems from the fact that we have assumed the matrices to be Hermitean, which prevents the eigenvalues from moving freely in the complex plane, and creating dense regions where the resolvent is no longer analytic. However, the proof does not really make use of the Hermiticity of the matrices, and the generalization to non-Hermitean matrices might provide some useful insight into these more complicated matrix models.

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